# Semianalytical Structural Sensitivity Formulation in Boundary Elements

Sunil Saigal*
*Carnegie Mellon University, Pittsburgh, Pennsylvania*
R. Aithal† and J. H. Kane‡
*Worcester Polytechnic Institute, Worcester, Massachusetts*

**A semianalytical formulation that combines the implicit differentiation of the discretized boundary integral equations and a univariate perturbation, forward-difference approximation technique is developed for the computation of structural design sensitivities. Two-dimensional plane and axisymmetric elastic continua, including body forces of the gravity and centrifugal types, are considered. As a result of the forward-difference approximation employed, the numerical integration of a new class of fundamental solution sensitivity kernels is avoided; and no special treatment is required for the evaluation of singular integrals for the diagonal entries in the boundary-element sensitivity system matrices. Because of the implicit-differentiation step, the factorization of the perturbed system matrices is avoided. Numerical data are presented for the study of the effect of the design variable perturbation size on the accuracy of the solution. Timings and accuracy results for test problems, including both plane and axisymmetric designs, are given and compared to the corresponding results obtained from a full-analytical sensitivity analysis. The present semianalytical method is computationally more efficient compared to the full-analytical implicit-differentiation approach for the same order of accuracy, for appropriate choices of design variable perturbation step sizes.**

## Introduction

THE derivatives of system response with respect to design parameters, termed design sensitivities, have been recognized to have a variety of uses in engineering research,[1] including structural optimization. Of the numerous papers published on design sensitivity analysis, most have been restricted to trusses or frameworks, and their applications to continuum structures are relatively rare.[2] Shape optimization, which may involve configuration parameters, nodal locations, boundary shapes, and overall topology[3] as design variables, thus constitutes a more complicated design optimization problem than in the case of member size selection. Because of this complexity, the most basic and straightforward approach to sensitivity analysis has been the finite-difference technique.[4,5] When used in conjunction with the finite-element method (FEM), this technique is computationally slow, its accuracy must be verified by convergence checks,[6] and the calculations are sensitive to small errors in initial data.[2] Analytical methods that reduce the computational efforts and provide exact derivatives have also been developed,[7,8] but they lack the ease of implementation and generality.[9] Prasad and Emerson[10] presented a semianalytical method for calculating derivatives that has the generality and programming ease of the finite-difference method along with the efficiency and accuracy of the analytical methods.

The boundary-element method (BEM) has recently been investigated for the computation of structural design sensitivities. The advantages of the BEM for sensitivity calculations are given in Ref. 11. A survey of sensitivity formulations in the BEM for structural, heat-transfer, aerodynamic, and other applications may be found in Ref. 12. The finite-difference approach in the BEM has been given by Wu.[13] The implicit-differentiation approach in the BEM, given by Kane and Saigal[14] for discontinuous plane elements, by Saigal et al.[15] for continuous plane elements, and by Saigal et al.[16] for axisymmetric elements, is an analytical approach that provides accurate sensitivity results. Another analytical approach based on direct differentiation in the BEM has been given by Barone and Yang.[17] The present paper presents a semianalytical sensitivity formulation in the BEM that strategically exploits the attractive features of the analytical implicit-differentiation approach and the forward-difference approximation, respectively. The implicit-differentiation approach[14-16] requires the numerical integration of a new class of fundamental solution sensitivity kernels, which exhibit a singular behavior similar to that of the fundamental solution elasticity kernels.[18] Successful implementation of this approach requires significant program development effort and careful analysis of the behavior of the new quantities during numerical integration. The present approach, however, obviates the need to formulate and perform the numerical integration of such new quantities. The forward-difference approximation is employed to obtain all matrix and vector sensitivities, which allows the existing boundary-element analysis codes to be utilized without significant modification by using input data associated with perturbed models.

The formulation of the semianalytical structural sensitivity analysis is presented. Timings and accuracy results for a range of test problems are given and compared to results obtained from a full-analytical sensitivity analysis capability. A discussion of strategies to exploit the substantial matrix sparsity that is generally present in sensitivity analysis is included, along with some general guidelines regarding the selection of design variable perturbation step sizes.

## Semianalytical Design Sensitivity Formulation

### Conventional Boundary Element Formulation

This section provides a brief summary of the boundary-element method to introduce notation. A detailed description may be found in, for instance, the text by Banerjee and Butterfield.[18]

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*Assistant Professor, Department of Civil Engineering. Associate Member AIAA.

†Graduate Research Assistant.

‡Assistant Professor, Mechanical Engineering Department.

The governing differential equations for the deformation of a homogeneous isotropic body with body forces $\psi_i$ are given by

$$(\lambda + \mu) \frac{\partial^2 u_j}{\partial x_i \partial x_j} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \psi_i = 0 \tag{1}$$

where $\lambda$ and $\mu$ are Lame's constants, and $u_i$ is the displacement vector referred to a Cartesian coordinate system $x_i$. The solution for Eq. (1) can be represented as

$$u_i = u_i^c + u_i^p \tag{2}$$

where $u_i^c$ and $u_i^p$ are the complementary solution and particular integral, respectively. By using the reciprocal work theorem on the Kelvin singular solution for a point load in infinite space and the complementary solution, the well-known Somigliana's identity is obtained in terms of the displacements $u^c$. Discretizing the boundary into a number of boundary elements and using Eq. (2) reduces the Somigliana's identity into the matrix form

$$[G]\{t - t^p\} - [F]\{u - u^p\} = \{0\} \tag{3}$$

Placing all of the known quantities on the right-hand side, we get

$$[A]\{x\} = \{b\} + [G]\{t^p\} - [F]\{u^p\} \tag{4}$$

where $\{b\}$ is a vector obtained from known tractions and displacements, $[A]$ a coefficient matrix, and $\{x\}$ a vector consisting of unknown tractions and displacements. The solution of the unknown vector $\{x\}$ in Eq. (4) requires the factorization of the coefficient matrix $[A]$. Equation (4) is valid for both plane and axisymmetric problems using appropriate kernel functions and coordinate definitions for each case. These details may be found, for example, in Ref. 18.

#### Analytical Implicit-Differentiation Sensitivity Formulation

The geometric sensitivity for the boundary of a solid object is determined through a small perturbation in the design variable $X_L$, with respect to which the sensitivity analysis is desired. The new geometry due to this perturbation is then obtained, and the geometric sensitivities are computed by applying a forward-difference relation.

For the implicit-differentiation formulation, a partial derivative of the discretized boundary integral relation in Eq. (3) is first performed and is given as

$$\begin{aligned}[G]_{,L}\{t\} + [G]\{t\}_{,L} &= [G]_{,L}\{t^p\} + [G]\{t^p\}_{,L} \\ &+ [F]_{,L}\{u\} + [F]\{u\}_{,L} - [F]_{,L}\{u^p\} - [F]\{u^p\}_{,L}\end{aligned} \tag{5}$$

Equation (5), without the particular integral contribution due to body forces, was given by Kane[19], and this equation was also independently obtained by Wu,[13] without the contribution due to body forces.

The expressions for the particular solutions $\{u^p\}$ and $\{t^p\}$ are given by Pape and Banerjee[20] for two-dimensional elastostatics and by Henry et al.[21] for axisymmetric cases. The sensitivities for these matrices required in Eq. (5) can be obtained by taking their partial derivatives with respect to the design variable and are given in Ref. 22. The sensitivities of the system matrices $[G]$ and $[F]$ are also required in Eq. (5). In the full-analytical formulation developed in Refs. 14–16, these sensitivities are evaluated through the numerical integration of terms involving a new class of fundamental solution sensitivity kernel functions. These integrals exhibit a singular behavior, and special treatment is required for the evaluation of the singular terms. A successful implementation thus requires significant program development effort and a careful analysis of the behavior of these new quantities.

It is noted that $u_{i,L} = 0$ when $u_i$ is specified and that $t_{i,L} = 0$ when $t_i$ is specified for any $i$. Using these conditions in Eq. (5) and placing all known quantities on the right-hand side or directly taking the partial derivative of Eq. (4) with respect to $X_L$ gives

$$[A]\{X\}_{,L} = \{b\}_{,L} - [A]_{,L}\{X\} + \{f\}_{,L} \tag{6}$$

where

$$\{f\}_{,L} = [G]_{,L}\{t^p\} + [G]\{t^p\}_{,L} - [F]_{,L}\{u^p\} - [F]\{u^p\}_{,L}$$

A significant advantage of the implicit-differentiation approach is that the coefficient matrix $[A]$ in the sensitivity equation (6) is the same as that in the analysis equation (4). The factorization of this matrix performed for the solution of Eq. (4) can thus be saved and reused for the solution of Eq. (6). The factorization of the new perturbed matrices is thus not required, leading to significant savings in computational effort.

#### Forward-Difference Approximation for System Matrix Sensitivities

The evaluation of the sensitivity matrices $[G]_{,L}$, $[F]_{,L}$, etc. is time-consuming and requires special treatment, as mentioned previously. The semianalytical approach presented here avoids the use of numerical integration for the computation of these sensitivity matrices. In this approach, a forward-difference approximation is used instead to obtain these matrices. A Taylor's series expansion of the derivative matrices gives

$$[F]_{,L} = \frac{[F(X_L + \Delta X_L)] - [F(X_L)]}{\Delta X_L} \tag{7}$$

and similarly for $[G]_{,L}$. The higher-order terms have been neglected in Eq. 7. This may lead to truncation errors in the evaluation of the derivatives for large values of step size, $\Delta X_L$. The truncation error is proportional to the step size as described in Ref. 5. In Eq. (7), the matrix $[F(X_L)]$ is obtained based on the initial design. The design variable $X_L$ is perturbed next by an amount $\Delta X_L$, and a new geometry is created for the continua. The matrix $[F(X_L + \Delta X_L)]$ is then computed based on this new geometry. The sensitivity matrix $[F]_{,L}$ can now be computed using Eq. 7. This procedure avoids the calculation of derivatives of quantities such as the kernel functions, Jacobian, and coordinate parameters. Because the computation of these derivative quantities is computationally burdensome, the semianalytical approach has significant computational economy compared to the full-analytical approach.

Another compelling aspect of the semianalytical sensitivity analysis is the ease with which it allows additional features to be treated for sensitivity computations. For example, loadings such as those due to gravity and rotation can be treated by computing the vector $\{f\}_{,L}$ by the forward-difference relation given as

$$\{f\}_{,L} = \frac{\{f(X_L + \Delta X_L)\} - \{f(X_L)\}}{\Delta X_L} \tag{8}$$

The new derivative expressions for the particular integrals and the related theoretical steps need not then be derived or computed. These features of the semianalytical approach allow easy extension of the existing boundary element analysis codes to include sensitivity computation capabilities.

#### Solution Procedure and Recovery of Boundary Stress Sensitivities

The boundary-element stress analysis of the initial design is first performed using Eq. (4), yielding the unknown displacements and tractions. The design variable $X_L$ is perturbed to obtain the new design, and the geometric sensitivities ($x_{,L}$, $y_{,L}$, etc.) are determined using a forward-difference relationship. The matrices $[G]_{,L}$, $[F]_{,L}$, $\{u^p\}_{,L}$, and $\{t^p\}_{,L}$ are determined using the forward-difference approximations shown in Eq. (7)

and (8). These matrices are substituted in Eq. (6), along with the other known quantities or quantities obtained previously through the solution of Eq. (4). The solution of Eq. (6) then yields the unknown displacement and traction sensitivities. To determine the sensitivities of the remaining stress components on the boundary, the following relationships are employed.

Two-dimensional plane sensitivity recovery:

$$\sigma_{11,L} = t_{1,L}, \qquad \sigma_{12,L} = t_{2,L}$$

$$\sigma_{22,L} = \frac{\nu}{1-\nu}\sigma_{11,L} + \frac{E}{1-\nu^2}\sigma_{22,L} \tag{9}$$

where

$$e_{22,L} = \frac{1}{J}u_{2,\xi L} - \frac{J^2}{J_{,L}}u_{2,\xi}$$

and where $\xi$ is the isoparametric coordinate direction, $J$ is the Jacobian, and $t_{1,L}$ and $t_{2,L}$ are traction sensitivities obtained from the solution of Eq. (6); $u_{2,\xi}$ and $u_{2,\xi L}$ are determined by using the interpolation functions; $E$ and $\nu$ are the modulus of elasticity and Poisson's ratio, respectively.

Axisymmetric sensitivity recovery:

$$\sigma_{11,L} = t_{1,L}, \qquad \sigma_{12,L} = t_{2,L}$$

$$\sigma_{22,L} = \frac{\nu}{1-\nu}\sigma_{11,L} + \frac{E}{1-\nu^2}(e_{22,L} + \nu e_{\theta\theta,L})$$

$$\sigma_{\theta\theta,L} = Ee_{\theta\theta,L} + \nu(\sigma_{11,L}r + \sigma_{22,L}) \tag{10}$$

where

$$e_{22,L} = \frac{1}{J}u_{2,\xi L} - \frac{J_{,L}}{J^2}u_{2,\xi}$$

$$e_{\theta\theta,L} = \frac{1}{r^2}[u_{r,L}r - u_r r_{,L}]$$

in which $r$ and $\theta$ are the radial and circumferential coordinate parameters, respectively. The remaining stress sensitivities are then obtained using Eqs. (9) and (10) for plane and axisymmetric cases, respectively. These evaluations do not require any additional numerical integration and are directly obtained.

## Numerical Results

The semianalytical procedure was used in computing the design sensitivities for both plane and axisymmetric problems. Examples with body forces of the gravity and centrifugal types are also presented. All computations were carried out on a RIDGE 3200 computer with a UNIX operating system.

### Simply Supported Beam Under Uniformly Distributed Load

A simply supported beam under a uniformly distributed load of 100 psi was analyzed. The geometric data for the beam are shown in Fig. 1; the material properties are: modulus of elasticity, $E = 30 \times 10^6$ psi, and Poisson's ratio $\nu = 0.3$; and the depth of the beam, $b$, was the design variable. The depth was varied uniformly across the entire span of the beam. Because of symmetry, only a half of the beam was modeled using 30 quadratic boundary elements, 5 along each of the vertical edges and 10 along each of the horizontal edges. The results were obtained using both continuous and discontinuous elements, respectively. Analytical sensitivity solutions for this case can be obtained by taking the material derivatives of the response solutions given by Timoshenko and Goodier.[23]

The study of the influence of the design variable perturbation step size on the percentage error in exact sensitivity results is given in Table 1 for both continuous and discontinuous elements, respectively. These results through low percentage errors indicate a good agreement between exact and computed sensitivity values for appropriate sizes of the perturbation step. It was also observed that the formulation for continuous elements is more stable with changes in step size compared to that for discontinuous elements. Similar studies on other example problems show that the results in Table 1 are typical of the behavior of sensitivity computations with changes in perturbation step size. Unless stated otherwise, a design variable perturbation of 0.1% was used in all results presented subsequently.

The semianalytical approach presented in this paper simplifies the computational effort compared to that required for the full-analytical approach given earlier.[14-16] A comparison of the accuracy performance of these two approaches is given in Table 2. For an appropriate selection of the design variable perturbation step size, the semianalytical approach is competitive from an accuracy standpoint. The numerical integration step for the computation of matrices $[A]_{,L}$ and $[B]_{,L}$ required 7.08 CPU s for the semianalytical approach compared to 13.58 CPU s required for the full-analytical approach. The semianalytical integration step was thus twice as fast as its full-analytical counterpart for only one design variable used in this problem. For more practical examples with numerous design variables, significant computational economy can be achieved using the present semianalytical approach.

**Table 1 Percentage errors with step size in semianalytical sensitivity calculations for a simply supported beam**

| | Location B | | | Location A | | |
|---|---|---|---|---|---|---|
| | Error, % | | | | | |
| Step size, % | $u_{,b}$ | $v_{,b}$ | $\sigma_{xx,b}$ | $u_{,b}$ | $v_{,b}$ | $\sigma_{xx,b}$ |
| | Continuous elements | | | | | |
| 1.00 | 0.97 | 0.62 | 0.96 | — | 0.96 | 0.81 |
| 0.75 | 0.73 | 0.52 | 0.75 | 0.67 | 0.54 | 0.60 |
| 0.50 | 0.50 | 0.41 | 0.53 | 0.47 | 0.43 | 0.89 |
| 0.25 | 0.13 | 0.30 | 0.32 | 0.24 | 0.32 | 0.18 |
| 0.10 | 0.11 | 0.24 | 0.19 | 0.12 | 0.24 | 0.05 |
| 0.075 | 0.09 | 0.23 | 0.17 | 0.10 | 0.23 | 0.03 |
| 0.050 | 0.06 | 0.22 | 0.15 | 0.08 | 0.22 | 0.01 |
| 0.025 | 0.052 | 0.214 | 0.137 | 0.067 | 0.216 | −0.004 |
| 0.010 | 0.0446 | 0.2109 | 0.1195 | 0.0546 | 0.2089 | −0.0167 |
| 0.001 | 0.0434 | 0.2026 | 0.1135 | 0.0471 | 0.2041 | −0.0250 |
| 0.0001 | 0.0434 | 0.2036 | 0.1135 | 0.0471 | 0.2041 | −0.0250 |
| $10^{-5}$ | 0.0434 | 0.2046 | 0.1135 | 0.0458 | 0.2041 | −0.0250 |
| $10^{-6}$ | 0.0434 | 0.2046 | 0.1135 | 0.0458 | 0.2041 | −0.0250 |
| | Discontinuous elements | | | | | |
| 0.1 | −0.361 | −0.216 | −0.059 | 0.366 | −0.222 | −0.350 |
| 0.01 | −0.277 | −0.125 | −0.077 | −0.291 | −0.135 | −0.279 |
| 0.0025 | −3.672 | −0.241 | −0.077 | −0.388 | −0.254 | −0.346 |
| 0.0010 | −4.805 | −2.527 | −3.309 | −4.852 | 4.740 | −4.899 |

### Rectangular Plate with an Elliptical Hole Under Tension

The sensitivity results for an infinite rectangular plate with an elliptical hole that experiences stress concentration due to uniaxial tension were obtained by Barone and Yang.[17] An analytical solution for this problem was also given in Ref. 17. The material properties used are: $E = 3 \times 10^7$ psi and $\nu = 0.3$. The major axis half-length $a$ was the design variable. This example was studied to include a problem with curved boundaries and a pronounced stress concentration.

A rectangular region of dimensions 800 × 800 in. was modeled using 33 equally spaced elements on each side. This region adequately simulates the infinite plate for a minor axis half-length, $b = 1$ in. Two different values of the major axis half-length, $a = 4$ and 8 in., respectively, were studied to observe the effect of severe stress raisers on the accuracy of sensitivity results. For both of these cases, the entire plate was modeled without exploiting the quarter-symmetry to provide a model similar to Ref. 17. The results for an 80-element mesh and a

**Table 2 Comparison of semianalytical and full-analytical sensitivity calculations for a simply supported beam**

| | Location B | | | Location A | | |
|---|---|---|---|---|---|---|
| | Error, % | | | | | |
| | $u_{,b}$ | $v_{,b}$ | $\sigma_{xx,b}$ | $u_{,b}$ | $v_{,b}$ | $\sigma_{xx,b}$ |
| SA-C[a] | 0.0434 | 0.2036 | 0.1135 | 0.0458 | 0.2036 | 0.1135 |
| FA-C[b] | 0.0409 | 0.2036 | 0.1135 | 0.0458 | 0.2041 | −0.0167 |
| SA-D[a] | −0.2776 | −0.1254 | 0.0776 | −0.0291 | −0.1356 | 0.2799 |
| FA-d[b] | −1.5503 | −1.3920 | −1.2370 | −2.1697 | −1.2848 | −1.3952 |

[a]Semianalytical approach, continuous element mesh. [b]Full-analytical approach, discontinuous element mesh.

**Table 3 Elliptical hole in an infinite rectangular plate. Semianalytical sensitivity results at major and minor axis locations**

| | Analytical | Aspect ratio $a/b = 4$, 80-element mesh | Aspect ratio $a/b = 4$, 100-element mesh | Aspect ratio $a/b = 8$, 80-element mesh | Aspect ratio $a/b = 8$, 100-element mesh |
|---|---|---|---|---|---|
| Location A | | | | | |
| $u_{,a} \times 10^6$ | −3.0333 | −3.0344 | −3.0344 | −3.0394 | −3.0409 |
| $\sigma_{yy,a}$ | 200.00 | 199.56 | 199.71 | 198.67 | 199.4 |
| Location B | | | | | |
| $v_{,a} \times 10^6$ | 6.0667 | 6.0631 | 6.0632 | 6.0709 | 6.0709 |
| $\sigma_{xx,a}$ | 0.0 | 0.0443 | −0.0016 | −0.0137 | −0.0139 |

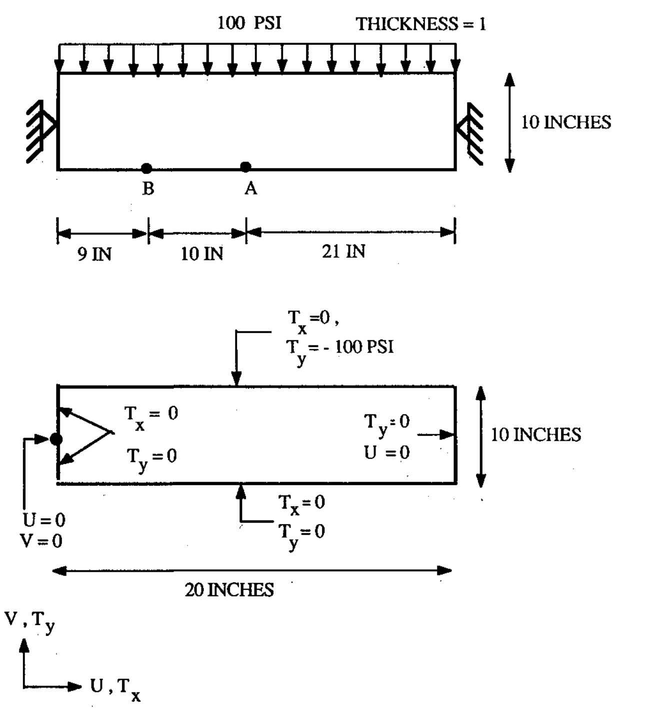


**Fig. 1 Simply supported beam under uniformly distributed load.**

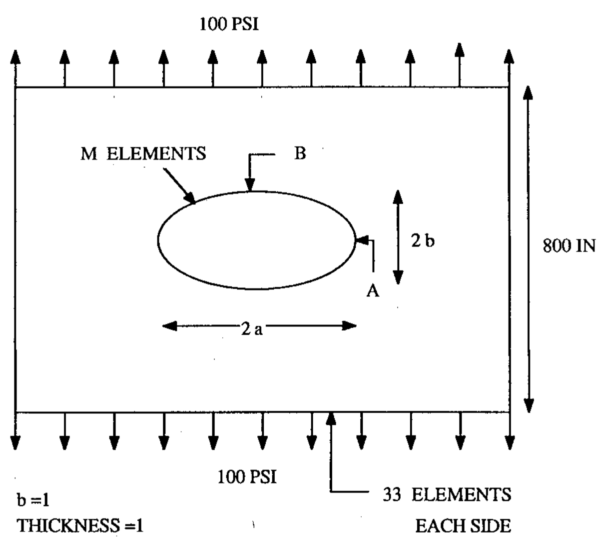


**Fig. 2 Rectangular plate with an elliptical hole under uniaxial tension.**

100-element mesh, respectively, of quadratic boundary elements for the ellipse are given in Table 3. The analytical sensitivities are also shown in Table 3 for comparison. The displacement and stress sensitivities at the respective boundary points *A* and *B*, shown in Fig. 2, only were given to avoid large amounts of data. The sensitivities obtained by using the present approach compare well with the analytical results. These sensitivities also depict the same order of accuracy as obtained in the full-analytical sensitivity results given in Ref. 15. The present approach also performs well for both values of stress concentrations in this example involving curved continuum geometry.

### Rectangular Strip Under Gravity Loading

The semianalytical approach was also applied to compute the structural sensitivities of designs under body forces. A hanging rectangular strip subjected to gravity loading due to its own weight was studied. The geometric dimensions for the strip are shown in Fig. 3; the length of the strip, *L*, was the design variable, and the material properties are $E = 10^7$ psi, $\nu = 0.3$, and mass density $\rho = 6$ lbm/in.$^3$ The analytical solution for the response of this design is given in Ref. 23, from which the analytical sensitivity expressions are obtained as follows:

$$u_{,L} = \frac{-\nu\rho g x y_{,L}}{E}, \qquad v_{,L} = \frac{\rho g y y_{,L}}{E} - \frac{\rho g L}{E}, \qquad \sigma_{yy,L} = \rho g y_{,L}$$

where $g$ ( = 386.4 in./s$^2$) is the acceleration due to gravity.

The rectangular strip was modeled using 50 boundary elements, 20 elements for each of the vertical edges, and 5 elements for each of the horizontal edges. The results for displacement and stress sensitivities for respective locations A–D in Fig. 3 are given in Table 4. The analytical solutions obtained from the sensitivity expressions just cited and those obtained using the present semianalytical formulation are shown in this table for accuracy comparisons. A good agreement is seen, and the results clearly demonstrate the viability of the semianalytical approach in the formation of the vector due to body forces, $\{f\}_{,L}$, in Eq. (10) using the forward-difference approximation.

### Rotating Solid Sphere

All of the examples considered up to now involve two-dimensional plane elasticity cases. The results for the semianalytical formulation for axisymmetric design are now presented. A solid sphere that rotates about its diametrical axis with a constant angular velocity of 10 rad/s was considered. The sphere has a radius of 10 in., and the material properties are $E = 3 \times 10^7$ psi, $\nu = 0.3$, and $\rho = 6$ lbm/in.$^3$ The analytical solution for this case is given by Love,[24] from which the analytical sensitivity solution can be obtained by taking appropriate material derivatives. The radius of the sphere was the design variable for this example.

Because of symmetry of the radial cross section, only a half of this section was modeled using quadratic, axisymmetric boundary elements. The effect of centrifugal loading resulting from the angular rotation of the sphere was taken into account. Two meshes consisting of 10 and 20 elements, respectively, were used. In each discretization, one half of the elements were used to model the straight radial boundary and the other half to model the curved boundary shown in Fig. 4. The displacement- and stress-sensitivity results using the present semianalytical approach are shown in Table 5, along with the exact analytical results for comparison. It is noted that the kernel functions given by Bakr[25] for the load point lying on the axis of rotational symmetry were used in the present study and that the load point was not moved a small distance away from the axis[19] for such cases. A good agreement of results was seen in Table 3, further demonstrating the accuracy of the semianalytical approach.

### Axially Loaded Notched Cylindrical Bar

A notched bar subjected to a uniaxial tension of 1000 psi was examined. The bar is shown in Fig. 5, and the sensitivity of axial stress at the stress concentration location $A$ with respect to the notch radius $R_c$ was determined. The material properties are: $E = 3 \times 10^7$ psi and $\nu = 0.3$. The full-analytical implicit-differentiation sensitivity solution for this example was given earlier by Saigal et al.[16]

Because of double symmetry of the radial section of the bar, a quarter of the bar was modeled using 21, 42, and 48 quadratic boundary elements, respectively, much as with the discretization in Ref. 16. The maximum stress concentration occurs in the bar at location A, Fig. 5. The sensitivity of displacement and axial stress at this location were obtained using the three respective discretizations. These results, along with the full-analytical implicit-differentiation results,[16] are shown in Table 6. The sensitivity results from the semianalytical approach agree very closely with those obtained from the full-analytical approach. Also, the sensitivity results appear to converge in a manner similar to the analysis results shown in Table 6.

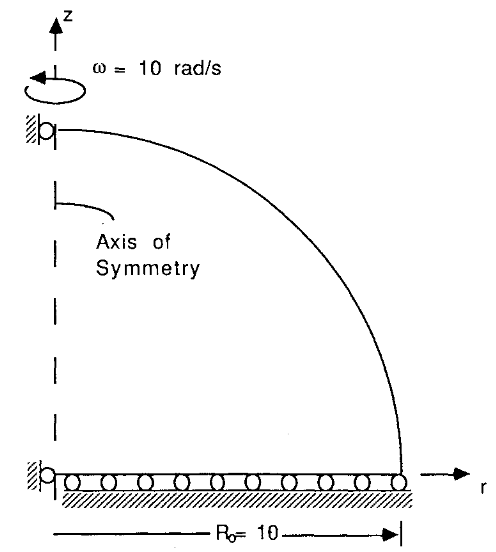


Fig. 4 Solid sphere rotating about its axis.

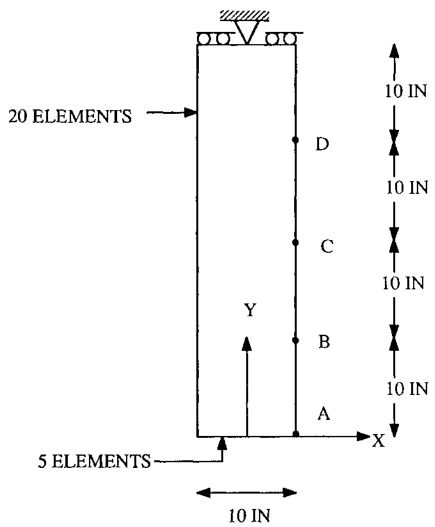


Fig. 3 Rectangular strip under self-weight.

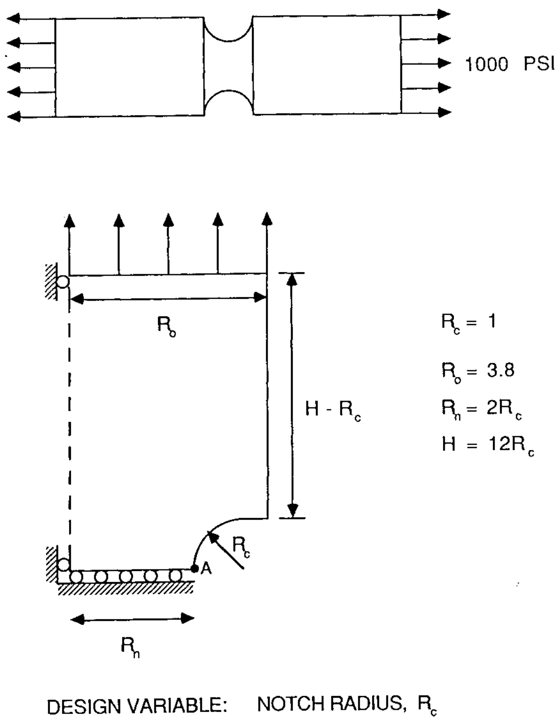


Fig. 5 Notched cylindrical bar under axial tension.

Table 4 Semianalytical sensitivities for a rectangular strip under gravity loading

| Location | Analytical | | | Present study | | |
|---|---|---|---|---|---|---|
| | $u_{,L} \times 10^5$ | $v_{,L} \times 10^3$ | $\sigma_{xx,L}$ | $u_{,L} \times 10^5$ | $v_{,L} \times 10^3$ | $\sigma_{xx,L}$ |
| A | 0.0 | 9.2739 | 0.0 | −0.0213 | 9.2757 | −0.6 |
| B | 0.8694 | 8.6950 | −579.6 | 0.8694 | 8.6950 | −580.8 |
| C | 1.7388 | 6.9551 | −1159.2 | 1.7392 | 6.9553 | −1159.8 |
| D | 2.6082 | 7.0569 | −1738.8 | 2.5979 | 4.0567 | −1741.3 |

Table 5 Sensitivity results for a solid sphere under centrifugal loading using the semianalytical approach

| Radius, in. | Sensitivities: Displacement, $\times 10^4$ | | | Stresses, $\times 10^2$ | | |
|---|---|---|---|---|---|---|
| | Analytical | 10 element model | 20 element model | Analytical | 10 element model | 20 element model |
| 0.00 | 0.00000 | 0.0000 | 0.0000 | 3.02521 | 2.8807 | 2.9290 |
| 1.00 | 1.71137 | 1.7109 | 1.7114 | 2.96471 | 2.9583 | 2.9597 |
| 2.00 | 3.38106 | 3.3801 | 3.3811 | 2.78319 | 2.7665 | 2.7838 |
| 3.00 | 4.96736 | 4.9659 | 4.9675 | 2.48067 | 2.4690 | 2.4817 |
| 4.00 | 6.42861 | 6.4267 | 6.4288 | 2.05714 | 2.0525 | 2.0583 |
| 5.00 | 7.72311 | 7.7208 | 7.7233 | 1.51261 | 1.5059 | 1.5138 |
| 6.00 | 8.80917 | 8.8064 | 8.8093 | 0.84706 | 0.8487 | 0.8483 |
| 7.00 | 9.64511 | 9.6418 | 9.6452 | 0.06050 | 0.0592 | 0.0618 |
| 8.00 | 10.1892 | 10.185 | 10.189 | −0.84706 | −0.8499 | −0.8458 |
| 9.00 | 10.3999 | 10.395 | 10.400 | −1.87563 | −1.8529 | −1.8735 |
| 10.00 | 10.2353 | 10.233 | 10.236 | −3.02521 | −3.0242 | −3.0022 |

Table 6 Semianalytical vs full-analytical sensitivities for a notched bar under tension

| No. of elements | $u_r$ | $u_{r,L} \times 10^5$ SA[a] | $u_{r,L} \times 10^5$ ID[b] | $\sigma_{zz}$ | $\sigma_{zz,L} \times 10^{-3}$ SA[a] | $\sigma_{zz,L} \times 10^{-3}$ ID[b] |
|---|---|---|---|---|---|---|
| 21 | −2.3065 | 3.7651 | 3.7623 | 6.2735 | 3.0866 | 3.0871 |
| 42 | −2.3063 | 3.7775 | 3.7747 | 6.3546 | 2.8874 | 2.8860 |
| 84 | −2.3060 | 3.7794 | 3.7766 | 6.3792 | 2.8206 | 2.8162 |

[a]Present semianalytical approach. [b]Implicit-differentiation approach.

## Conclusions

A semianalytical formulation for the computation of design sensitivities in the boundary-element method is developed. This approach does not require the definition and numerical integration of a new class of fundamental solution sensitivity kernel functions, is easy to implement, and allows for the use, with minor modifications, of existing boundary-element analysis codes. The key step in this procedure is the strategic application of a forward-difference approximation for certain matrix and vector sensitivities in an otherwise completely analytical implicit-differentiation formulation for sensitivity analysis. Numerical data are given to study the stability of the semianalytical approach with the design variable perturbation step size. An appropriate choice of this step size is very important and, for complicated problems, this choice may become as difficult as is sometimes the case when the standard finite-difference method is applied to this class of problems. Timings and accuracy results for a wide range of examples involving both plane and axisymmetric designs are given and compared to corresponding full-analytical results. The present approach is approximately twice as efficient computationally as the full-analytical approach, with the same order of accuracy for an appropriate choice of the design variable perturbation step size. The results were presented for both continuous and discontinuous elements. The continuous elements appear to depict a more stable behavior over a wider range of design variable perturbation step size. The present approach is straightforward and efficient and may, in addition, be used for checking new analytical sensitivity formulations.

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